2-7 Rates, Ratios, and Proportions

Objectives
Write and use ratios, rates, and unit rates.

Write and solve proportions.

Vocabulary
ratio	proportion
rate	cross products
scale	scale drawing
unit rate	scale model
conversion factor	dimensional analysis

Why learn this?
Ratios and proportions are used to draw accurate maps. (See Example 5.)

A **ratio** is a comparison of two quantities by division. The ratio of a to b can be written $a:b$ or $\frac{a}{b}$, where $b \neq 0$. Ratios that name the same comparison are said to be *equivalent*.

A statement that two ratios are equivalent, such as $\frac{1}{12} = \frac{2}{24}$, is called a **proportion**.

EXAMPLE 1 Using Ratios

The ratio of faculty members to students at a college is 1:15. There are 675 students. How many faculty members are there?

$$\frac{\text{faculty}}{\text{students}} \rightarrow \frac{1}{15}$$ 	*Write a ratio comparing faculty to students.*

$$\frac{1}{15} = \frac{x}{675}$$ 	*Write a proportion. Let x be the number of faculty members.*

$$675\left(\frac{x}{675}\right) = 675\left(\frac{1}{15}\right)$$ 	*Since x is divided by 675, multiply both sides of the equation by 675.*

$$x = 45$$

There are 45 faculty members.

Reading Math

Read the proportion $\frac{1}{15} = \frac{x}{675}$ as "1 is to 15 as x is to 675."

CHECK IT OUT!
1. The ratio of games won to games lost for a baseball team is $3:2$. The team won 18 games. How many games did the team lose?

A **rate** is a ratio of two quantities with different units, such as $\frac{34 \text{ mi}}{2 \text{ gal}}$. Rates are usually written as *unit rates*. A **unit rate** is a rate with a second quantity of 1 unit, such as $\frac{17 \text{ mi}}{1 \text{ gal}}$, or 17 mi/gal. You can convert any rate to a unit rate.

EXAMPLE 2 Finding Unit Rates

Takeru Kobayashi of Japan ate 53.5 hot dogs in 12 minutes to win a contest. Find the unit rate in hot dogs per minute. Round to the nearest hundredth.

$$\frac{53.5}{12} = \frac{x}{1}$$ 	*Write a proportion to find an equivalent ratio with a second quantity of 1.*

$$4.46 \approx x$$ 	*Divide on the left side to find x.*

The unit rate is approximately 4.46 hot dogs per minute.

CHECK IT OUT!
2. Cory earns $52.50 in 7 hours. Find the unit rate in dollars per hour.

Dimensional analysis is a process that uses rates to convert measurements from one unit to another. A rate such as $\frac{12 \text{ in.}}{1 \text{ ft}}$, in which the two quantities are equal but use different units, is called a **conversion factor**. To convert from one set of units to another, multiply by a conversion factor.

EXAMPLE 3 Using Dimensional Analysis

A A large adult male human has about 12 pints of blood. Use dimensional analysis to convert this quantity to gallons.

Step 1 Convert pints to quarts.

$12 \text{ pt} \cdot \dfrac{1 \text{ qt}}{2 \text{ pt}}$ *Multiply by a conversion factor whose first quantity is quarts and whose second quantity is pints.*

6 qt

12 pints is 6 quarts.

Step 2 Convert quarts to gallons.

$6 \text{ qt} \cdot \dfrac{1 \text{ gal}}{4 \text{ qt}}$ *Multiply by a conversion factor whose first quantity is gallons and whose second quantity is quarts.*

$\dfrac{6}{4} \text{ gal} = 1\dfrac{1}{2} \text{ gal}$

A large adult male human has about $1\frac{1}{2}$ gallons of blood.

B The dwarf sea horse *Hippocampus zosterae* swims at a rate of 52.68 feet per hour. Use dimensional analysis to convert this speed to inches per minute.

Use the conversion factor $\frac{12 \text{ in.}}{1 \text{ ft}}$ to convert feet to inches, and use the conversion factor $\frac{1 \text{ h}}{60 \text{ min}}$ to convert hours to minutes.

$$\frac{52.68 \text{ ft}}{1 \text{ h}} \cdot \frac{12 \text{ in.}}{1 \text{ ft}} \cdot \frac{1 \text{ h}}{60 \text{ min}} = \frac{10.536 \text{ in.}}{1 \text{ min}}$$

The speed is 10.536 inches per minute.

Check that the answer is reasonable. The answer is about 10 in./min.

- There are **60** min in 1 h, so **10** in./min is **60**(**10**) = 600 in./h.

- There are **12** in. in 1 ft, so **600** in./h is $\frac{600}{12}$ = 50 ft/h. This is close to the rate given in the problem, 52.68 ft/h.

Hippocampus zosterae

CHECK IT OUT!

3. A cyclist travels 56 miles in 4 hours. Use dimensional analysis to convert the cyclist's speed to feet per second. Round your answer to the nearest tenth, and show that your answer is reasonable.

In the proportion $\frac{a}{b} = \frac{c}{d}$, the products $a \cdot d$ and $b \cdot c$ are called **cross products**. You can solve a proportion for a missing value by using the Cross Products Property.

Know it! Note

Cross Products Property

WORDS	NUMBERS	ALGEBRA
In a proportion, cross products are equal.	$\dfrac{2}{3} \diagdown\!\!\!\!\diagup \dfrac{4}{6}$ $2 \cdot 6 = 3 \cdot 4$	If $\dfrac{a}{b} \diagdown\!\!\!\!\diagup \dfrac{c}{d}$ and $b \neq 0$ and $d \neq 0$, then $ad = bc$.

Helpful Hint

In Example 3A Step 1, "pt" appears to divide out, leaving "qt," as the unit. Use this strategy of "dividing out" units when using dimensional analysis.

EXAMPLE 4

Solving Proportions

Solve each proportion.

A $\dfrac{5}{9} = \dfrac{3}{w}$

$\dfrac{5}{9} \bowtie \dfrac{3}{w}$

$5(w) = 9(3)$ *Use cross products.*

$5w = 27$

$\dfrac{5w}{5} = \dfrac{27}{5}$ *Divide both sides by 5.*

$w = \dfrac{27}{5}$

B $\dfrac{8}{x + 10} = \dfrac{1}{12}$

$\dfrac{8}{x + 10} \bowtie \dfrac{1}{12}$

$8(12) = 1(x + 10)$ *Use cross products.*

$96 = x + 10$

$\underline{-10 \qquad -10}$ *Subtract 10 from both sides.*

$86 = x$

CHECK IT OUT!

Solve each proportion.

4a. $\dfrac{-5}{2} = \dfrac{y}{8}$

4b. $\dfrac{g + 3}{5} = \dfrac{7}{4}$

A **scale** is a ratio between two sets of measurements, such as 1 in : 5 mi. A **scale drawing** or **scale model** uses a scale to represent an object as smaller or larger than the actual object. A map is an example of a scale drawing.

EXAMPLE 5

Scale Drawings and Scale Models

A On the map, the distance from Chicago to Evanston is 0.625 in. What is the actual distance?

$\dfrac{\text{map}}{\text{actual}} \rightarrow \dfrac{1 \text{ in.}}{18 \text{ mi}}$ *Write the scale as a fraction.*

$\dfrac{1}{18} \bowtie \dfrac{0.625}{x}$ *Let x be the actual distance.*

$x \cdot 1 = 18(0.625)$ *Use cross products to solve.*

$x = 11.25$

The actual distance is 11.25 mi.

B The actual distance between North Chicago and Waukegan is 4 mi. What is this distance on the map? Round to the nearest tenth.

$\dfrac{\text{map}}{\text{actual}} \rightarrow \dfrac{1 \text{ in.}}{18 \text{ mi}}$ *Write the scale as a fraction.*

$\dfrac{1}{18} \bowtie \dfrac{x}{4}$ *Let x be the distance on the map.*

$4 = 18x$ *Use cross products to solve the proportion.*

$\dfrac{4}{18} = \dfrac{18x}{18}$ *Since x is multiplied by 18, divide both sides by 18 to undo the multiplication.*

$0.2 \approx x$ *Round to the nearest tenth.*
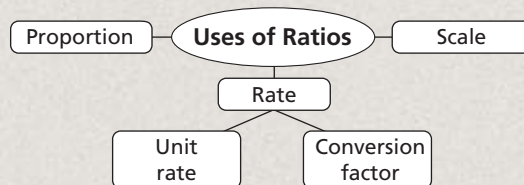
The distance on the map is about 0.2 in.

Reading Math

A scale written without units, such as 32 : 1, means that 32 units of any measure correspond to 1 unit of that same measure.

Waukegan
North Chicago
Highland Park
Evanston
1 in : 18 mi
Chicago

CHECK IT OUT!

5. A scale model of a human heart is 16 ft long. The scale is 32 : 1. How many inches long is the actual heart it represents?
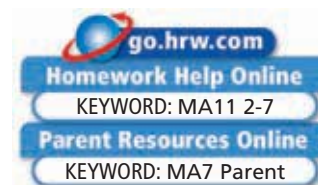
THINK AND DISCUSS

1. Explain two ways to solve the proportion $\frac{t}{4} = \frac{3}{5}$.

2. How could you show that the answer to Example 5A is reasonable?

3. **GET ORGANIZED** Copy and complete the graphic organizer. In each box, write an example of each use of ratios.

```
Proportion —— Uses of Ratios —— Scale
                    |
                  Rate
                 /    \
             Unit      Conversion
             rate       factor
```

Exercises

GUIDED PRACTICE

1. Vocabulary What does it mean when two ratios form a *proportion*?

SEE EXAMPLE **1**
p. 120

2. The ratio of the sale price of a jacket to the original price is $3:4$. The original price is $64. What is the sale price?

3. Chemistry The ratio of hydrogen atoms to oxygen atoms in water is $2:1$. If an amount of water contains 341 trillion atoms of oxygen, how many hydrogen atoms are there?

SEE EXAMPLE **2**
p. 120

4. A computer's fan rotates 2000 times in 40 seconds. Find the unit rate in rotations per second.

5. Twelve cows produce 224,988 pounds of milk. Find the unit rate in pounds per cow.

6. A yellow jacket can fly 4.5 meters in 9 seconds. Find the unit rate in meters per second.

SEE EXAMPLE **3**
p. 121

7. Lydia wrote $4\frac{1}{2}$ pages of her science report in one hour. What was her writing rate in pages per minute?

8. A model airplane flies 18 feet in 2 seconds. What is the airplane's speed in miles per hour? Round your answer to the nearest hundredth.

9. A vehicle uses 1 tablespoon of gasoline to drive 125 yards. How many miles can the vehicle travel per gallon? Round your answer to the nearest mile. (*Hint:* There are 256 tablespoons in a gallon.)

SEE EXAMPLE **4**
p. 122

Solve each proportion.

10. $\frac{3}{z} = \frac{1}{8}$

11. $\frac{x}{3} = \frac{1}{5}$

12. $\frac{b}{4} = \frac{3}{2}$

13. $\frac{f+3}{12} = \frac{7}{2}$

14. $\frac{-1}{5} = \frac{3}{2d}$

15. $\frac{3}{14} = \frac{s-2}{21}$

16. $\frac{-4}{9} = \frac{7}{x}$

17. $\frac{3}{s-2} = \frac{1}{7}$

18. $\frac{10}{h} = \frac{52}{13}$

19. **Archaeology** Stonehenge II in Hunt, Texas, is a scale model of the ancient construction in Wiltshire, England. The scale of the model to the original is 3:5. The Altar Stone of the original construction is 4.9 meters tall. Write and solve a proportion to find the height of the model of the Altar Stone.

Alfred Sheppard, one of the builders of Stonehenge II.

PRACTICE AND PROBLEM SOLVING

20. **Gardening** The ratio of the height of a bonsai ficus tree to the height of a full-size ficus tree is 1:9. The bonsai ficus is 6 inches tall. What is the height of a full-size ficus?

21. **Manufacturing** At one factory, the ratio of defective light bulbs produced to total light bulbs produced is about 3:500. How many light bulbs are expected to be defective when 12,000 are produced?

22. Four gallons of gasoline weigh 25 pounds. Find the unit rate in pounds per gallon.

23. Fifteen ounces of gold cost $6058.50. Find the unit rate in dollars per ounce.

24. **Biology** The tropical giant bamboo can grow 11.9 feet in 3 days. What is this rate of growth in inches per hour? Round your answer to the nearest hundredth, and show that your answer is reasonable.

25. **Transportation** The maximum speed of the Tupolev Tu-144 airliner is 694 m/s. What is this speed in kilometers per hour?

Solve each proportion.

26. $\dfrac{v}{6} = \dfrac{1}{2}$ **27.** $\dfrac{2}{5} = \dfrac{4}{y}$ **28.** $\dfrac{2}{h} = \dfrac{-5}{6}$ **29.** $\dfrac{3}{10} = \dfrac{b+7}{20}$

30. $\dfrac{5t}{9} = \dfrac{1}{2}$ **31.** $\dfrac{2}{3} = \dfrac{6}{q-4}$ **32.** $\dfrac{x}{8} = \dfrac{7.5}{20}$ **33.** $\dfrac{3}{k} = \dfrac{45}{18}$

34. $\dfrac{6}{a} = \dfrac{15}{17}$ **35.** $\dfrac{9}{2} = \dfrac{5}{x+1}$ **36.** $\dfrac{3}{5} = \dfrac{x}{100}$ **37.** $\dfrac{38}{19} = \dfrac{n-5}{20}$

38. **Science** The image shows a dust mite as seen under a microscope. The scale of the drawing to the dust mite is 100:1. Use a ruler to measure the length of the dust mite in the image in millimeters. What is the actual length of the dust mite?

39. **Finance** On a certain day, the exchange rate was 60 U.S. dollars for 50 euro. How many U.S. dollars were 70 euro worth that day? Show that your answer is reasonable.

40. **Environmental Science** An environmental scientist wants to estimate the number of carp in a pond. He captures 100 carp, tags all of them, and releases them. A week later, he captures 85 carp and records how many have tags. His results are shown in the table. Write and solve a proportion to estimate the number of carp in the pond.

Status	Number Captured
Tagged	20
Not tagged	65

41. /// **ERROR ANALYSIS** /// Below is a bonus question that appeared on an algebra test and a student's response.

> The ratio of junior varsity members to varsity members on the track team is 3:5. There are 24 members on the team. Write a proportion to find the number of junior varsity members.
>
> $$\frac{3}{5} = \frac{x}{24}$$

The student did not receive the bonus points. Why is this proportion incorrect?

42. **Sports** The table shows world record times for women's races of different distances.

World Records (Women)	
Distance (m)	Time (s)
100	10.5
200	21.3
800	113.3
5000	864.7

  a. Find the speed in meters per second for each race. Round your answers to the nearest hundredth.

  b. Which race has the fastest speed? the slowest?

  c. **Critical Thinking** Give a possible reason why the speeds are different.

43. **Entertainment** Lynn, Faith, and Jeremy are film animators. In one 8-hour day, Lynn rendered 203 frames, Faith rendered 216 frames, and Jeremy rendered 227 frames. How many more frames per hour did Faith render than Lynn did?

Solve each proportion.

44. $\dfrac{x-1}{3} = \dfrac{x+1}{5}$

45. $\dfrac{m}{3} = \dfrac{m+4}{7}$

46. $\dfrac{1}{x-3} = \dfrac{3}{x-5}$

47. $\dfrac{a}{2} = \dfrac{a-4}{30}$

48. $\dfrac{3}{2y} = \dfrac{16}{y+2}$

49. $\dfrac{n+3}{5} = \dfrac{n-1}{2}$

50. $\dfrac{1}{y} = \dfrac{1}{6y-1}$

51. $\dfrac{2}{n} = \dfrac{4}{n+3}$

52. $\dfrac{5t-3}{-2} = \dfrac{t+3}{2}$

53. $\dfrac{3}{d+3} = \dfrac{4}{d+12}$

54. $\dfrac{3x+5}{14} = \dfrac{x}{3}$

55. $\dfrac{5}{2n} = \dfrac{8}{3n-24}$

56. **Decorating** A particular shade of paint is made by mixing 5 parts red paint with 7 parts blue paint. To make this shade, Shannon mixed 12 quarts of blue paint with 8 quarts of red paint. Did Shannon mix the correct shade? Explain.

57. **Write About It** Give three examples of proportions. How do you know they are proportions? Then give three nonexamples of proportions. How do you know they are not proportions?

MULTI-STEP TEST PREP

58. This problem will prepare you for the Multi-Step Test Prep on page 152.

  a. Marcus is shopping for a new jacket. He finds one with a price tag of $120. Above the rack is a sign that says that he can take off $\frac{1}{5}$. Find out how much Marcus can deduct from the price of the jacket.

  b. What price will Marcus pay for the jacket?

  c. Copy the model below. Complete it by placing numerical values on top and the corresponding fractional parts below.

$0	?	$48	?	?	$120
0	$\frac{1}{5}$	?	?	$\frac{4}{5}$	1
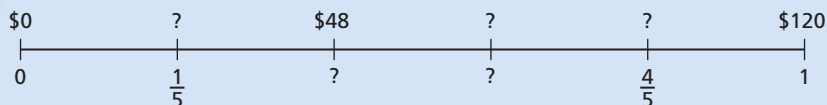
  d. Explain how this model shows proportional relationships.

59. One day the U.S. dollar was worth approximately 100 yen. An exchange of 2500 yen was made that day. What was the value of the exchange in dollars?

   Ⓐ $25   Ⓑ $400   Ⓒ $2500   Ⓓ $40,000

60. Brett walks at a speed of 4 miles per hour. He walks for 20 minutes in a straight line at this rate. Approximately what distance does Brett walk?

   Ⓕ 0.06 miles   Ⓖ 1.3 miles   Ⓗ 5 miles   Ⓙ 80 miles

61. A shampoo company conducted a survey and found that 3 out of 8 people use their brand of shampoo. Which proportion could be used to find the expected number of users n in a city of 75,000 people?

   Ⓐ $\dfrac{3}{8} = \dfrac{75,000}{n}$ Ⓑ $\dfrac{3}{75,000} = \dfrac{n}{8}$ Ⓒ $\dfrac{8}{3} = \dfrac{n}{75,000}$ Ⓓ $\dfrac{3}{8} = \dfrac{n}{75,000}$

62. A statue is 3 feet tall. The display case for a model of the statue can fit a model that is no more than 9 inches tall. Which of the scales below allows for the tallest model of the statue that will fit in the display case?

   Ⓕ 2:1   Ⓖ 1:1   Ⓗ 1:3   Ⓙ 1:4

CHALLENGE AND EXTEND

63. **Geometry** Complementary angles are two angles whose measures add up to 90°. The ratio of the measures of two complementary angles is 4:5. What are the measures of the angles?

64. A customer wanted 24 feet of rope. The clerk at the hardware store used what she thought was a yardstick to measure the rope, but the yardstick was actually 2 inches too short. How many inches were missing from the customer's piece of rope?

65. **Population** The population density of Jackson, Mississippi, is 672.2 people per square kilometer. What is the population density in people per square meter? Show that your answer is reasonable. (*Hint:* There are 1000 meters in 1 kilometer. How many square meters are in 1 square kilometer?)

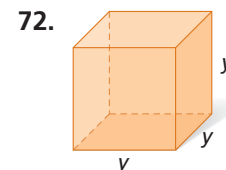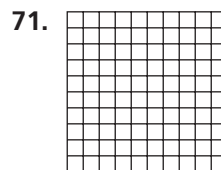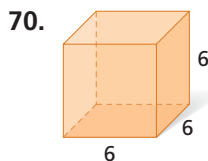## SPIRAL REVIEW

Evaluate each expression. *(Lesson 1-4)*

66. 8^2 **67.** $(-3)^3$ **68.** $(-3)^2$ **69.** $-\left(\dfrac{1}{2}\right)^5$

Write the power represented by each geometric model. *(Lesson 1-4)*

70.

71.

72.

Solve each equation. Check your answer. *(Lesson 2-4)*

73. $2x - 12 = 5x + 3$ **74.** $3a - 4 = 6 - 7a$ **75.** $3x - 4 = 2x + 4$

Solve for the indicated variable. *(Lesson 2-5)*

76. $y = mx + b$ for b **77.** $PV = nRT$ for V **78.** $A = \dfrac{1}{2}bh$ for h