7-5 Rational Exponents

Objective
Evaluate and simplify expressions containing rational exponents.

Vocabulary
index

Why learn this?
You can use rational exponents to find the number of Calories animals need to consume each day to maintain health. (See Example 3.)

Recall that the radical symbol $\sqrt{\ }$ is used to indicate roots. The **index** is the small number to the left of the radical symbol that tells which root to take. For example, $\sqrt[3]{\ }$ represents a cube root. Since $2^3 = 2 \cdot 2 \cdot 2 = 8$, $\sqrt[3]{8} = 2$.

Another way to write nth roots is by using exponents that are fractions. For example, for $b > 1$, suppose $\sqrt{b} = b^k$.

$$\begin{aligned} \sqrt{b} &= b^k \\ \left(\sqrt{b}\right)^2 &= \left(b^k\right)^2 && \textit{Square both sides.} \\ b^1 &= b^{2k} && \textit{Power of a Power Property} \\ 1 &= 2k && \textit{If } b^m = b^n\textit{, then } m = n. \\ \frac{1}{2} &= k && \textit{Divide both sides by 2.} \end{aligned}$$

So for all $b > 1$, $\sqrt{b} = b^{\frac{1}{2}}$.

Helpful Hint
When $b = 0$, $\sqrt[n]{b} = 0$.
When $b = 1$, $\sqrt[n]{b} = 1$.

Know it! Note

Definition of $b^{\frac{1}{n}}$

WORDS	NUMBERS	ALGEBRA
A number raised to the power of $\frac{1}{n}$ is equal to the nth root of that number.	$3^{\frac{1}{2}} = \sqrt{3}$ $5^{\frac{1}{4}} = \sqrt[4]{5}$ $2^{\frac{1}{7}} = \sqrt[7]{2}$	If $b > 1$ and n is an integer, where $n \geq 2$, then $b^{\frac{1}{n}} = \sqrt[n]{b}$. $b^{\frac{1}{2}} = \sqrt{b}$, $b^{\frac{1}{3}} = \sqrt[3]{b}$, $b^{\frac{1}{4}} = \sqrt[4]{b}$, and so on.

EXAMPLE 1 Simplifying $b^{\frac{1}{n}}$

Simplify each expression.

A $125^{\frac{1}{3}}$

$125^{\frac{1}{3}} = \sqrt[3]{125} = \sqrt[3]{5^3}$ *Use the definition of $b^{\frac{1}{n}}$.*
$= 5$

B $64^{\frac{1}{6}} + 25^{\frac{1}{2}}$

$64^{\frac{1}{6}} + 25^{\frac{1}{2}} = \sqrt[6]{64} + \sqrt{25}$ *Use the definition of $b^{\frac{1}{n}}$.*
$= \sqrt[6]{2^6} + \sqrt{5^2}$
$= 2 + 5 = 7$

Remember!
$\sqrt{\ }$ is equivalent to $\sqrt[2]{\ }$. See Lesson 1-5.

Simplify each expression.

1a. $81^{\frac{1}{4}}$ **1b.** $121^{\frac{1}{2}} + 256^{\frac{1}{4}}$

A fractional exponent can have a numerator other than 1, as in the expression $b^{\frac{2}{3}}$. You can write the exponent as a product in two different ways.

$$b^{\frac{2}{3}} = b^{\frac{1}{3}\cdot 2} \qquad b^{\frac{2}{3}} = b^{2\cdot\frac{1}{3}}$$
$$= \left(b^{\frac{1}{3}}\right)^2 \quad \textit{Power of a Power Property} \qquad = \left(b^2\right)^{\frac{1}{3}}$$
$$= \left(\sqrt[3]{b}\right)^2 \quad \textit{Definition of } b^{\frac{1}{n}} \qquad = \sqrt[3]{b^2}$$

Know it! Note

Definition of $b^{\frac{m}{n}}$

WORDS	NUMBERS	ALGEBRA
A number raised to the power of $\frac{m}{n}$ is equal to the nth root of the number raised to the mth power.	$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$ $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$	If $b > 1$ and m and n are integers, where $m \geq 1$ and $n \geq 2$, then $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$.

EXAMPLE 2 Simplifying Expressions with Fractional Exponents

Simplify each expression.

A $216^{\frac{2}{3}}$

$$216^{\frac{2}{3}} = \left(\sqrt[3]{216}\right)^2 \quad \textit{Definition of } b^{\frac{m}{n}}$$
$$= \left(\sqrt[3]{6^3}\right)^2$$
$$= (6)^2 = 36$$

B $32^{\frac{4}{5}}$

$$32^{\frac{4}{5}} = \left(\sqrt[5]{32}\right)^4$$
$$= \left(\sqrt[5]{2^5}\right)^4$$
$$= (2)^4 = 16$$

Simplify each expression.

2a. $16^{\frac{3}{4}}$ **2b.** $1^{\frac{2}{5}}$ **2c.** $27^{\frac{4}{3}}$

EXAMPLE 3 *Biology Application*

The approximate number of Calories C that an animal needs each day is given by $C = 72m^{\frac{3}{4}}$, where m is the animal's mass in kilograms. Find the number of Calories that a 16 kg dog needs each day.

$$C = 72m^{\frac{3}{4}}$$
$$= 72(16)^{\frac{3}{4}} \quad \textit{Substitute 16 for m.}$$
$$= 72 \cdot \left(\sqrt[4]{16}\right)^3 \quad \textit{Definition of } b^{\frac{m}{n}}$$
$$= 72 \cdot \left(\sqrt[4]{2^4}\right)^3$$
$$= 72 \cdot (2)^3$$
$$= 72 \cdot 8 = 576$$

The dog needs 576 Calories per day to maintain health.

3. Find the number of Calories that an 81 kg panda needs each day.

Remember that $\sqrt{\ }$ always indicates a nonnegative square root. When you simplify variable expressions that contain $\sqrt{\ }$, such as $\sqrt{x^2}$, the answer cannot be negative. But x may be negative. Therefore you simplify $\sqrt{x^2}$ as $|x|$ to ensure the answer is nonnegative.

When x is...	and n is...	x^n is...	and $\sqrt[n]{x^n}$ is...
Positive	Even	Positive	Positive
Negative	Even	Positive	Positive
Positive	Odd	Positive	Positive
Negative	Odd	Negative	Negative

When n is even, you must simplify $\sqrt[n]{x^n}$ to $|x|$, because you do not know whether x is positive or negative. When n is odd, simplify $\sqrt[n]{x^n}$ to x.

EXAMPLE 4 Using Properties of Exponents to Simplify Expressions

Simplify. All variables represent nonnegative numbers.

A $\sqrt[3]{x^9y^3}$

$$\sqrt[3]{x^9y^3} = (x^9y^3)^{\frac{1}{3}} \qquad \textit{Definition of } b^{\frac{1}{n}}$$
$$= (x^9)^{\frac{1}{3}} \cdot (y^3)^{\frac{1}{3}} \qquad \textit{Power of a Product Property}$$
$$= \left(x^{9\cdot\frac{1}{3}}\right) \cdot \left(y^{3\cdot\frac{1}{3}}\right) \qquad \textit{Power of a Power Property}$$
$$= (x^3) \cdot (y^1) = x^3y \qquad \textit{Simplify exponents.}$$

B $\left(x^2y^{\frac{1}{2}}\right)^4 \sqrt[3]{y^3}$

$$\left(x^2y^{\frac{1}{2}}\right)^4 \sqrt[3]{y^3} = \left(x^2y^{\frac{1}{2}}\right)^4 \cdot y \qquad \sqrt[3]{y^3} = y$$
$$= (x^{2\cdot4}) \cdot \left(y^{\frac{1}{2}\cdot4}\right) \cdot y \qquad \textit{Power of a Product Property}$$
$$= (x^8) \cdot (y^2) \cdot y \qquad \textit{Simplify exponents.}$$
$$= x^8 \cdot y^{2+1} = x^8y^3 \qquad \textit{Product of Powers Property}$$

Helpful Hint

When you are told that all variables represent nonnegative numbers, you do not need to use absolute values in your answers.

CHECK IT OUT! **Simplify. All variables represent nonnegative numbers.**

4a. $\sqrt[4]{x^4y^{12}}$

4b. $\dfrac{\left(xy^{\frac{1}{2}}\right)^2}{\sqrt[5]{x^5}}$

THINK AND DISCUSS

1. Explain how to find the value of $\left(\sqrt[10]{25}\right)^5$.

Know it! Note

2. GET ORGANIZED Copy and complete the graphic organizer. In each cell, provide the definition and a numerical example of each type of exponent.

Exponent	Definition	Numerical Example
$b^{\frac{1}{n}}$		
$b^{\frac{m}{n}}$		

7-5 Exercises

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GUIDED PRACTICE

1. **Vocabulary** In the expression $\sqrt[5]{3x}$, what is the *index*?

Simplify each expression.

SEE EXAMPLE 1 p. 488

2. $8^{\frac{1}{3}}$ 3. $16^{\frac{1}{2}}$ 4. $0^{\frac{1}{6}}$ 5. $27^{\frac{1}{3}}$
6. $81^{\frac{1}{2}}$ 7. $216^{\frac{1}{3}}$ 8. $1^{\frac{1}{9}}$ 9. $625^{\frac{1}{4}}$
10. $36^{\frac{1}{2}} + 1^{\frac{1}{3}}$ 11. $8^{\frac{1}{3}} + 64^{\frac{1}{2}}$ 12. $81^{\frac{1}{4}} + 8^{\frac{1}{3}}$ 13. $25^{\frac{1}{2}} - 1^{\frac{1}{4}}$

SEE EXAMPLE 2 p. 489

14. $81^{\frac{3}{4}}$ 15. $8^{\frac{5}{3}}$ 16. $125^{\frac{2}{3}}$ 17. $25^{\frac{3}{2}}$
18. $36^{\frac{3}{2}}$ 19. $64^{\frac{4}{3}}$ 20. $1^{\frac{3}{4}}$ 21. $0^{\frac{2}{3}}$

SEE EXAMPLE 3 p. 489

22. **Geometry** Given a square with area a, you can use the formula $P = 4a^{\frac{1}{2}}$ to find the perimeter P of the square. Find the perimeter of a square that has an area of 64 m^2.

SEE EXAMPLE 4 p. 490

Simplify. All variables represent nonnegative numbers.

23. $\sqrt{x^4y^2}$ 24. $\sqrt[4]{z^4}$ 25. $\sqrt{x^6y^6}$ 26. $\sqrt[3]{a^{12}b^6}$
27. $\left(a^{\frac{1}{2}}\right)^2\sqrt{a^2}$ 28. $\left(x^{\frac{1}{3}}\right)^6\sqrt[4]{y^4}$ 29. $\dfrac{\left(z^{\frac{1}{3}}\right)^3}{\sqrt{z^2}}$ 30. $\dfrac{\sqrt[3]{x^6y^9}}{x^2}$

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
31–42	1
43–50	2
51	3
52–59	4

Extra Practice
Skills Practice p. S17
Application Practice p. S34

Simplify each expression.

31. $100^{\frac{1}{2}}$ 32. $1^{\frac{1}{5}}$ 33. $512^{\frac{1}{3}}$ 34. $729^{\frac{1}{2}}$
35. $32^{\frac{1}{5}}$ 36. $196^{\frac{1}{2}}$ 37. $256^{\frac{1}{8}}$ 38. $400^{\frac{1}{2}}$
39. $125^{\frac{1}{3}} + 81^{\frac{1}{2}}$ 40. $25^{\frac{1}{2}} - 81^{\frac{1}{4}}$ 41. $121^{\frac{1}{2}} - 243^{\frac{1}{5}}$ 42. $256^{\frac{1}{4}} + 0^{\frac{1}{3}}$
43. $4^{\frac{3}{2}}$ 44. $27^{\frac{2}{3}}$ 45. $256^{\frac{3}{4}}$ 46. $64^{\frac{5}{6}}$
47. $100^{\frac{3}{2}}$ 48. $1^{\frac{5}{3}}$ 49. $9^{\frac{5}{2}}$ 50. $243^{\frac{2}{5}}$

51. **Biology** Biologists use a formula to estimate the mass of a mammal's brain. For a mammal with a mass of m grams, the approximate mass B of the brain, also in grams, is given by $B = \frac{1}{8}m^{\frac{2}{3}}$. Find the approximate mass of the brain of a mouse that has a mass of 64 grams.

Simplify. All variables represent nonnegative numbers.

52. $\sqrt[3]{a^6c^9}$ 53. $\sqrt[3]{8m^3}$ 54. $\sqrt[4]{x^{16}y^4}$ 55. $\sqrt[3]{27x^6}$
56. $\left(x^{\frac{1}{2}}y^3\right)^2\sqrt{x^2}$ 57. $\left(a^2b^4\right)^{\frac{1}{2}}\sqrt[3]{b^6}$ 58. $\dfrac{\sqrt[3]{x^6y^6}}{yx^2}$ 59. $\dfrac{\left(a^2b^{\frac{1}{2}}\right)^4}{\sqrt{b^2}}$

Fill in the boxes to make each statement true.

60. $256^{\square} = 4$ 61. $\square^{\frac{1}{5}} = 1$ 62. $225^{\square} = 15$ 63. $\square^{\frac{1}{6}} = 0$
64. $64^{\square} = 16$ 65. $\square^{\frac{3}{4}} = 125$ 66. $27^{\square} = 81$ 67. $36^{\square} = 216$

Simplify each expression.

68. $\left(\frac{81}{169}\right)^{\frac{1}{2}}$ **69.** $\left(\frac{8}{27}\right)^{\frac{1}{3}}$ **70.** $\left(\frac{256}{81}\right)^{\frac{1}{4}}$

71. $\left(\frac{1}{16}\right)^{\frac{1}{2}}$ **72.** $\left(\frac{9}{16}\right)^{\frac{3}{2}}$ **73.** $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

74. $\left(\frac{16}{81}\right)^{\frac{3}{4}}$ **75.** $\left(\frac{4}{49}\right)^{\frac{3}{2}}$ **76.** $\left(\frac{4}{25}\right)^{\frac{3}{2}}$

77. $\left(\frac{1}{81}\right)^{\frac{3}{4}}$ **78.** $\left(\frac{27}{64}\right)^{\frac{2}{3}}$ **79.** $\left(\frac{8}{125}\right)^{\frac{4}{3}}$

80. Multi-Step Scientists have found that the life span of a mammal living in captivity is related to the mammal's mass. The life span in years L can be approximated by the formula $L = 12m^{\frac{1}{5}}$, where m is the mammal's mass in kilograms. How much longer is the life span of a lion compared with that of a wolf?

Typical Mass of Mammals	
Mammal	**Mass (kg)**
Koala	8
Wolf	32
Lion	243
Giraffe	1024

81. Geometry Given a sphere with volume V, the formula $r = 0.62V^{\frac{1}{3}}$ may be used to approximate the sphere's radius r. Find the approximate radius of a sphere that has a volume of 27 in^3.

82. Critical Thinking Show that a number raised to the power $\frac{1}{3}$ is the same as the cube root of that number. (*Hint:* Use properties of exponents to find the cube of $b^{\frac{1}{3}}$. Then compare this with the cube of $\sqrt[3]{b}$. Use the fact that if two numbers have the same cube, then they are equal.)

83. Critical Thinking Compare $n^{\frac{2}{3}}$ and $n^{\frac{3}{2}}$ for values of n greater than 1. When simplifying each of these expressions, will the result be greater than n or less than n? Explain.

84. ERROR ANALYSIS Two students simplified $64^{\frac{3}{2}}$. Which solution is incorrect? Explain the error.

A

$64^{\frac{3}{2}} = \left(\sqrt[3]{64}\right)^2$
$= (4)^2$
$= 16$

B

$64^{\frac{3}{2}} = \left(\sqrt{64}\right)^3$
$= (8)^3$
$= 512$

MULTI-STEP TEST PREP

85. This problem will prepare you for the Multi-Step Test Prep on page 494.
You can estimate an object's distance in inches from a light source by using the formula $d = \left(0.8\frac{L}{B}\right)^{\frac{1}{2}}$, where L is the light's luminosity in lumens and B is the light's brightness in lumens per square inch.

a. Find an object's distance to a light source with a luminosity of 4000 lumens and a brightness of 32 lumens per square inch.

b. Suppose the brightness of this light source decreases to 8 lumens per square inch. How does the object's distance from the source change?

86. Write About It You can write $4^{\frac{3}{2}}$ as $4^{3 \cdot \frac{1}{2}}$ or as $4^{\frac{1}{2} \cdot 3}$. Use the Power of a Power Property to show that both expressions are equal. Is one method easier than the other? Explain.

TEST PREP

87. What is $9^{\frac{1}{2}} + 8^{\frac{1}{3}}$?

Ⓐ 4 Ⓑ 5 Ⓒ 6 Ⓓ 10

88. Which expression is equal to 8?

Ⓕ $4^{\frac{3}{2}}$ Ⓖ $16^{\frac{1}{2}}$ Ⓗ $32^{\frac{4}{5}}$ Ⓙ $64^{\frac{3}{2}}$

89. Which expression is equivalent to $\sqrt[3]{a^9b^3}$?

Ⓐ a^2b Ⓑ a^3 Ⓒ a^3b Ⓓ a^3b^3

90. Which of the following is NOT equal to $16^{\frac{3}{2}}$?

Ⓕ $\left(\sqrt{16}\right)^3$ Ⓖ 4^3 Ⓗ $\left(\sqrt[3]{16}\right)^2$ Ⓙ $\sqrt{16^3}$

CHALLENGE AND EXTEND

Use properties of exponents to simplify each expression.

91. $\left(a^{\frac{1}{3}}\right)\left(a^{\frac{1}{3}}\right)\left(a^{\frac{1}{3}}\right)$ **92.** $\left(x^{\frac{1}{2}}\right)^5\left(x^{\frac{3}{2}}\right)$ **93.** $\left(x^{\frac{1}{3}}\right)^4\left(x^5\right)^{\frac{1}{3}}$

You can use properties of exponents to help you solve equations. For example, to solve $x^3 = 64$, raise both sides to the $\frac{1}{3}$ power to get $\left(x^3\right)^{\frac{1}{3}} = 64^{\frac{1}{3}}$. Simplifying both sides gives $x = 4$. Use this method to solve each equation. Check your answer.

94. $y^5 = 32$ **95.** $27x^3 = 729$ **96.** $1 = \frac{1}{8}x^3$

97. Geometry The formula for the surface area of a sphere S in terms of its volume V is $S = (4\pi)^{\frac{1}{3}}(3V)^{\frac{2}{3}}$. What is the surface area of a sphere that has a volume of 36π cm^3? Leave the symbol π in your answer. What do you notice?

SPIRAL REVIEW

Solve each equation. *(Lesson 2-6)*

98. $|x + 6| = 2$ **99.** $|5x + 5| = 0$ **100.** $|2x - 1| = 3$

Solve each inequality and graph the solutions. *(Lesson 3-4)*

101. $3n + 5 < 14$ **102.** $4 \leq \frac{1}{2}x + 3$ **103.** $7 \geq 2y + 11$

Give the domain and range of each relation. Tell whether the relation is a function. Explain. *(Lesson 4-2)*

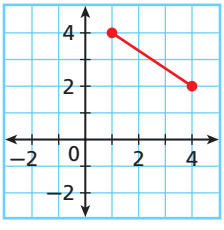
104. $\{(2, 3), (2, 4), (2, 5), (2, 6)\}$

105. $\{(-2, 0), (-1, 1), (0, 2), (1, 3)\}$

106.

x	y
5	2
7	2
9	2
11	2

107.



CHAPTER 7

MULTI-STEP TEST PREP

SECTION 7A

Exponents

I See the Light! The speed of light is the product of its frequency f and its wavelength w. In air, the speed of light is 3×10^8 m/s.

1. Write an equation for the relationship described above, and then solve this equation for frequency. Write this equation as an equation with w raised to a negative exponent.
2. Wavelengths of visible light range from 400 to 700 nanometers (10^{-9} meters). Use a graphing calculator and the relationship you found in Problem 1 to graph frequency as a function of wavelength. Sketch the graph with the axes clearly labeled. Describe your graph.
3. The speed of light in water is $\frac{3}{4}$ of its speed in air. Find the speed of light in water.
4. When light enters water, some colors bend more than others. How much the light bends depends on its wavelength. This is what creates a rainbow. The frequency of green light is about 5.9×10^{14} cycles per second. Find the wavelength of green light in water.
5. When light enters water, colors with shorter wavelengths bend more than colors with longer wavelengths. Violet light has a frequency of 7.5×10^{14} cycles per second, and red light has a frequency of 4.6×10^{14} cycles per second. Which of these colors of light will bend more when it enters water? Justify your answer.

Quiz for Lessons 7-1 Through 7-5

7-1 Integer Exponents

Evaluate each expression for the given value(s) of the variable(s).

1. t^{-6} for $t = 2$ **2.** n^{-3} for $n = -5$ **3.** $r^0 s^{-2}$ for $r = 8$ and $s = 10$

Simplify.

4. $5k^{-3}$ **5.** $\frac{x^4}{y^{-6}}$ **6.** $8f^{-4}\,g^0$ **7.** $\frac{a^{-3}}{b^{-2}}$

8. Measurement Metric units can be written in terms of a base unit. The table shows some of these equivalencies. Simplify each expression.

Selected Metric Prefixes					
Milli-	Centi-	Deci-	Deka-	Hecto-	Kilo-
10^{-3}	10^{-2}	10^{-1}	10^{1}	10^{2}	10^{3}

7-2 Powers of 10 and Scientific Notation

9. Find the value of 10^4.

10. Write 0.0000001 as a power of 10.

11. Write 100,000,000,000 as a power of 10.

12. Find the value of 82.1×10^4.

13. Measurement The lead in a mechanical pencil has a diameter of 0.5 mm. Write this number in scientific notation.

7-3 Multiplication Properties of Exponents

Simplify.

14. $2^2 \cdot 2^5$ **15.** $3^5 \cdot 3^{-3}$ **16.** $p^4 \cdot p^5$ **17.** $a^3 \cdot a^{-6} \cdot a^{-2}$

18. Biology A swarm of locusts was estimated to contain 2.8×10^{10} individual insects. If each locust weighs about 2.5 grams, how much did this entire swarm weigh? Write your answer in scientific notation.

Simplify.

19. $(3x^4)^3$ **20.** $(m^3n^2)^5$ **21.** $(-4d^7)^2$ **22.** $(cd^6)^3 \cdot (c^5d^2)^2$

7-4 Division Properties of Exponents

Simplify.

23. $\frac{6^9}{6^7}$ **24.** $\frac{12a^5}{3a^2}$ **25.** $\left(\frac{3}{5}\right)^3$ **26.** $\left(\frac{4p^3}{2pq^4}\right)^2$

Simplify each quotient and write the answer in scientific notation.

27. $(8 \times 10^9) \div (2 \times 10^6)$ **28.** $(3.5 \times 10^5) \div (7 \times 10^8)$ **29.** $(1 \times 10^4) \div (4 \times 10^4)$

7-5 Rational Exponents

Simplify each expression. All variables represent nonnegative numbers.

30. $81^{\frac{1}{2}}$ **31.** $125^{\frac{1}{3}}$ **32.** $4^{\frac{3}{2}}$ **33.** $0^{\frac{2}{9}}$

34. $\sqrt{x^8y^4}$ **35.** $\sqrt[3]{r^9}$ **36.** $\sqrt[6]{z^{12}}$ **37.** $\sqrt[3]{p^3q^{12}}$